Final Exam Review

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1. Gaussian Elimination and solutions to systems of equations.
   (i) Know how to apply Gaussian elimination to an augmented matrix to solve a system of linear equations (both non-homogeneous and homogeneous) and how to write the solution set.
   (ii) Know how to find the basic solutions, equivalently, a basis, for the solution space of of a homogeneous system of linear equations.
   (iii) Note that if A is an $m \times n$ matrix, then n equals rank of A plus the number of independent parameters needed to describe the solution space of the homogeneous system having A as the coefficient matrix, where the rank of A is the number of leading ones in the reduced row echelon form of A .

   Equivalently, the rank of A plus the dimension of the null space of A equals n.

2. Inverses and determinant

(i) Know how to find the inverse and the determinant of a square matrix.

(ii) For the inverse, this can be done using Gaussian Elimination on an augmented matrix of the form $\begin{bmatrix} A & | & I_n \end{bmatrix}$, until the left hand side is I_n. The right hand side will then be A^{-1}.

(iii) One should know the simple formula for the inverse of a 2×2 matrix using the determinant.

(iv) Note that If A is an $n \times n$ matrix that is invertible, then the solution to the system of equations $A \cdot \mathrm{X} = \mathrm{b}$ is given by $\mathrm{X} = A^{-1} \cdot \mathrm{b}$.

(v) Students should know how to calculate the determinant of a square matrix by expanding along any row or column, and also by using elementary row operations.

(vi) Students should know how to solve a system of equations using Cramer's Rule.

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Students should know the follow important fact: For an $n \times n$ matrix A, the following are equivalent:

(i) The system of equations $A \cdot \mathrm{X} = \mathrm{b}$ has a unique solution.

(ii) The matrix A is invertible.

(iii) The rank of A equals n.

(iv) The determinant of A is not zero.

(v) The columns of A are linearly independent.

(vi) The columns of A span $\mathbb{R}^n$.

(vii) The columns of A form a basis for $\mathbb{R}^n$.

It is important to note that items (v)-(vii) are not equivalent if A is not a square matrix.

3. Eigenvalues, eigenvectors, and diagonalizability of square matrices. Let A be an $n \times n$ matrix.

(i) The real number λ is an **eigenvalue** of A is there exists a **non-zero** vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$. In this case, v is an **eigenvector** associate to λ.

(ii) The eigenvalues of A are the roots of $c_A(x)$, the characteristic polynomial of A. $c_A(x) = \det[xI_n - A]$.

(iii) For a given eigenvalue λ, the λ-eigenvectors are the non-zero zero vectors in the null space of the matrix $\lambda I_n - A$. The basic solutions in this null space are **basic** λ-eigenvectors and form a **basis** for the eigenspace E_λ.

(iv) If A is an $n \times n$ matrix, then, by definition, A is **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP = D$, where D is an $n \times n$ diagonal matrix.

(v) If A is diagonalizable, the diagonal entries of the matrix D in (iv) are the eigenvalues of A.

(vi) Suppose $c_A(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$, then the eigenvalue λ_i has **multiplicity** e_i.

(vii) A is diagonalizable if and only if $c_A(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_r)^{e_r}$ and for each eigenvalue λ_i, e_i equals the dimension of E_{λ_i}.

In particular: A is diagonalizable if A has n distinct eigenvalues.

(viii) If A is diagonalizable, then the diagonalizing matrix P is obtained by taking the matrix whose columns are the collection of basic eigenvectors derived from A.

4. Applications of diagonalizability of square matrices. Suppose A is diagonalizable, with $P^{-1}AP = D$.

(i) $A = PDP^{-1}$, and therefore $A^n = PD^nP^{-1}$, for all $n \geq 1$.

(ii) For any square matrix B, e^B is the matrix given by the Taylor Series: $\sum_{n=0}^{\infty} \frac{1}{n!} B^n$.

(iii) If $D = \text{diag}(\lambda_1, \dots, \lambda_n)$, then $e^D = \text{diag}(e^{\lambda_1}, \dots, e^{\lambda_n})$.

(iv) For A diagonalizable, $e^A = Pe^DP^{-1}$.

(v) Solving recurrence relations: A sequence of non-negative numbers $a_0, a_1, a_2, \ldots, a_k, \ldots,$ is called a **linear recursion sequence of length two** if there are fixed integers α, β, c, d such that:

(i) $a_0 = \alpha$.
(ii) $a_1 = \beta$.
(iii) $a_{k+2} = c \cdot a_k + d \cdot a_{k+1}$, for all $k \geq 0$.

To find a closed form solution for a_k, let $v_k = \begin{bmatrix} a_k \\ a_{k+1} \end{bmatrix}$, and $A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$. Then $v_k = A^k \cdot v_0$, and a_k is the first coordinate of the vector v_k.

(vi) Solving systems of first order linear differential equations: Let $A = (a_{ij})$, be an $n \times n$ matrix. A system of first order linear differential equations is a system of equations of the form:

$$\begin{aligned} x_1'(t) &= a_{11}x_1(t) + \cdots + a_{1n}x_n(t) \\ x_2'(t) &= a_{21}x_1(t) + \cdots + a_{2n}x_n(t) \\ \vdots \quad &= \qquad\qquad \vdots \\ x_n'(t) &= a_{n1}x_1(t) + \cdots + a_{nn}x_n(t), \end{aligned}$$

where $x_i(t)$ is a real valued function of t. The numbers $x_1(0), \cdots, x_n(0)$ are called the *initial conditions* of the system.

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In matrix form, the system is given by the equation: $X'(t) = A \cdot X(t)$,

where $X(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ and $X'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$.

The solution to the system is given by: $X(t) = e^{At} \cdot X(0)$.

Here: $e^{At} = Pe^{Dt}P^{-1}$, where, $e^{Dt} = \text{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$ and P is the matrix diagonalizing A.

5. Spanning sets, linear independence and bases in Euclidean space. Let $v_1, \dots, v_r, w$ be columns vectors in $\mathbb{R}^n$. Let $A = [v_1 \ v_2 \cdots \ v_r]$. Then:

(i) w belongs to span$\{v_1, \dots, v_r\}$ if and only if the system of equations $A \cdot \mathrm{X} = w$ has a solution.

(ii) If $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ is a solution to $A \cdot \mathrm{X} = w$, then $w = \lambda_1 v_1 + \cdots + \lambda_r v_r$.

(iii) $v_1, \dots, v_r$ are linearly independent if and only if $A \cdot \mathrm{X} = \mathbf{0}$ has only the zero solution.

(iv) If $v_1, \dots, v_r$ are **not** linearly independent and $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ is a non-zero solution to $A \cdot \mathrm{X} = 0$, then

$$(*) \quad \lambda_1 v_1 + \cdots \lambda_r v_r = \mathbf{0}.$$

This means the vectors $v_1, \dots, v_r$ are linearly dependent, and thus redundant.

(v) One can use (*) to write some v_i in terms of the remaining v's. Upon doing so:

$$\text{span}\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r\} = \text{span}\{v_1, \ldots, v_r\}.$$

(vi) One may continue to eliminate redundant vectors from among the v_i's.

As soon as one one arrives at a linearly independent subset of $v_1, \ldots, v_r$, this set of vectors forms a basis for the original subspace $\text{span}\{v_1, \ldots, v_r\}$.

The number of elements in the basis is then the dimension of $\text{span}\{v_1, \ldots, v_r\}$.

(vii) To test if the n vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$ are linearly independent, or span $\mathbb{R}^n$, or form a basis for $\mathbb{R}^n$, it suffices to show that $\det[v_1 \ v_2 \ \cdots \ v_n] \neq 0$.

6. Orthogonalization.

(i) Given finitely many (linearly independent) vectors in $\mathbb{R}^n$ spanning a subspace U, know how to apply the Gram-Schimidt process to find : (a) An orthogonal basis for U and(b) An orthonormal basis for U.

(ii) Know how to use the dot product to write a vector $u \in U$ as a linear combination of an orthonormal basis.

(iii) Given a vector b in $\mathbb{R}^n$, know how to calculate its orthogonal projection p_Ub onto U.

(iv) Note that that p_Ub can be used to find the best approximation to a solution of a system of linear equations $A \cdot \mathrm{X} = \mathrm{b}$ that does not have a solution. In this case, if the columns of A are orthogonal, any solution z to the system $A \cdot \mathrm{X} = p_U\mathrm{b}$ will be a best approximation to a solution.

(v) Similarly, if z satisfies $A^t A \cdot \mathrm{z} = A^t \cdot \mathrm{b}$, then z is also a best approximation.

(vi) Given finitely many data points, know how to use this second method to find the line of best fit or the quadratic of best fit passing through the data points. (See Lecture 22).